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SOME PROPERTIES OF FAMILIES OF CONVEX CONES([1])

BY

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ABSTRACT. The purpose of this paper is to study properties of finite families of convex cones in n-dimensional Euclidean space R^n, whose members all have the origin as a common apex.

Of special interest are such families of convex cones in R^n which have the following property: Each member of the family is of dimension n, the intersection of any two members is at least $(n-1)$-dimensional, ..., the intersection of any n members is at least 1-dimensional and the intersection of all the members is the origin.

1. **Introduction.** The purpose of this paper is to study properties of families of convex cones in n-dimensional Euclidean space in R^n, whose members have the origin as a common apex.

For a set A in R^n, $\dim A$ denotes the dimension of the minimal flat containing A. For a family $A_T = \{A_i;\ i \in T\}$ of sets in R^n, $A(S)$ denotes $\bigcap\{A_i\colon i \in S\}$ and $\bar{A}(S) = A(T \setminus S)$. We use the convention that $A(\emptyset) = \bar{A}(T) = R^n$.

Unless stated otherwise a family is a finite family and a cone is a *convex cone with apex* 0.

Of special interest are nonempty families of cones in R^n which are nondegenerate in the following sense:

Each member of the family is of dimension n, the intersection of any two members of the family is of dimension $n-1$ at least, ..., the intersection of any n members of the family is of dimension 1 *at least and the intersection of all members of the family is the origin*. Such families are called *nondegenerate families* or *N.D.F.s*.

Equivalently: A family A_T of cones in R^n is an N.D.F. if $A(T) = \{0\}$ and $\dim A(S) \geq n - |S| + 1$ for any nonempty $S \subset T$. Three properties of N.D.F.s are given in Theorems 1–3:

THEOREM 1 (PERLES). *If A_T is an N.D.F. in R^n then A_T covers R^n (i.e., $\bigcup\{A_i\colon i \in T\} = R^n$).*

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THEOREM 2. *If A_T is an N.D.F. in R^n then for each $j \in T$ the intersection of all members of A_T excluding A_j is contained in $-A_j$ (i.e., $\bar{A}(\{j\}) \subset -A_j$).*

THEOREM 3. *If A_T is an N.D.F., M is a subset of T and $\bar{A}(M) = A(T \setminus M)$ contains an M-dimensional subspace then M is the empty set.*

Theorem 1 was originally proved by M. A. Perles (private communication). Perles' proof is algebraic in nature.

A geometric proof of Theorem 2 is given in §2.

It is possible to prove Theorems 1 and 3 using the same methods described in §2.

Let A_T be a family of cones in R^n. Subsets B of T for which $\bar{A}(B)$ is a subspace will be called *faces of A_T*. B will be called a *k-face of A_T* if B is a face of A_T and $\dim \bar{A}(B) = |B| - k$.

It is natural to ask the following question:

Given a family A_T of convex cones in R^n and given $\dim A(S)$ for each $S \subset T$, can we determine the subsets B of T which are faces of A_T?

In general, as easy examples can show, the answer is negative.

However, Theorem 1 yields a sufficient condition:

By Theorem 1, if $B \neq \emptyset$ and $\{\bar{A}(B) \cap A_i : i \in B\}$ is an N.D.F. in span $\bar{A}(B)$, then $\bigcup \{\bar{A}(B) \cap A_i : i \in B\} = \text{span } \bar{A}(B)$ and therefore $\bar{A}(B)$ is a subspace. Thus we have

THEOREM 4. *If $B = \emptyset$ and $A(T) = \{0\}$ or if the family $\{\bar{A}(B) \cap A_i : i \in B\}$ is an N.D.F. in* span $\bar{A}(B)$, *then B is a face of A_T.*

A subset B of T will be called a nondegenerate subset of T *relative to A_T or an N.D.S. of A_T* if $B = \emptyset$ and $A(T) = \{0\}$ or if the family $\{\bar{A}(B) \cap A_i : i \in B\}$ is an N.D.F. in span $\bar{A}(B)$.

Equivalently: a subset B of T will be called an *N.D.S.* of A_T if $B = \emptyset$ and $A(T) = \{0\}$ or if $\bar{A}(\emptyset) = A(T) = 0$ and $|S| - \dim \bar{A}(S) < |B| - \dim \bar{A}(B)$ for any proper subset S of B.

B will be called a *k-N.D.S. of A_T* if it is an N.D.S. of A_T and if $\dim \bar{A}(B) = |B| - k$.

If A_T is an N.D.F. then the sufficient condition in Theorem 4 is also a necessary condition:

THEOREM 5. *If A_T is an N.D.F. then a subset B of T is a [k-]face of A_T iff it is a [k-]N.D.S. of A_T.*

The proof of Theorem 5 is established in §3.

Theorem 5 yields an algorithm for finding all the faces of an N.D.F. when $\dim A(S)$ is known for all $S \subset T$.

ALGORITHM. If A_T is an N.D.F. then a subset B of T is a face of A_T iff

$|S| - \dim \bar{A}(S) < |B| - \dim \bar{A}(B)$ for any proper subset S of B.

In the last section we attempt to 'justify' nondegenerate families of convex cones by discussing some results which may be obtained by using N.D.F.s.

2. **Proof of Theorem 2.** The essence of the proof is the use of a suitable separating hyperplane and induction on n.

The following lemma will be used:

LEMMA 1. *If A and B are polyhedral cones in R^n, A is pointed and $A \cap B = 0$, then there is a hyperplane H which separates A and B and strictly supports A (i.e., A and B are on different sides of H and $H \cap A = \{0\}$).*

Let $A_T = \{A_i\colon i \in T\}$ be an N.D.F. in R^n. It is enough to prove that $\bigcup \{A_i\colon i \in T\} = R^n$ under the additional assumption that A_T is a family of *polyhedral cones*.

The proof is by induction on n and for fixed n by induction on $t = |T| - n$ (since A_T is an N.D.F., $t > 1$).

If $\bar{A}(j) = A(T \setminus \{j\}) = 0$, the proof is trivial (this includes the case $n = 0$).

Otherwise. Suppose $0 \neq x \in \bar{A}(j)$. We have to show that $-x \in A_j$. There are two cases to consider.

1. For each $i \in T \setminus \{j\}$, $\bar{A}(i) \neq \{0\}$. (This includes the case $t = 1$.)

If $\bar{A}(i)$ contains a line for each $i \in T \setminus \{j\}$ we would have $A_j = R^n$ (since A_j contains the sum of these lines, the lines are linearly independent ($A(T) = \{0\}$), and there are at least n lines).

This leads to $x \in A_j \cap \bar{A}(j) = A(T) = \{0\}$, a contradiction.

Suppose $i \in T \setminus \{j\}$ and $\bar{A}(i)$ is a pointed cone. Using Lemma 1, let H be a hyperplane which separates $\bar{A}(i)$ and A_i (both of them are polyhedral by our assumptions).

Define $A'_{T\setminus\{i\}}$, an N.D.F. in $R^{n-1} = H$ by

$$A'_l = A_l \cap H \quad \text{for } l \in T \setminus \{i\}.$$

(It is not difficult to prove that $A'_{T\setminus\{i\}}$ is an N.D.F.)

Let H^+ and H^- be the two closed halfspaces determined by H, and suppose that $\bar{A}(i) \subseteq H^+$. Since $x \in \bar{A}(j) \subset A_i \subset H^-$ and $\bar{A}(i) \cap \operatorname{int} H^+ \neq \emptyset$, there is a $y \in \bar{A}(i)$ such that $x + y \in H$. Therefore $x + y \in \bar{A}(\{i \cup j\}) \cap H = \bar{A}'(j)$.

By the induction assumption (on n), $-x - y \in A'_i \subset A_i$. Consequently, $-x = (-x - y) + y \in A_j$ ($y \in \bar{A}(i) \subset A_j$).

2. Suppose $i \in T \setminus \{j\}$ and $\bar{A}(i) = \{0\}$. Define $A'_{T\setminus\{i\}} = \{A'_f = A_f\colon f \in T \setminus \{i\}\}$. Then $A'_{T\setminus\{i\}}$ is an N.D.F. of $n + (t - 1)$ cones in R^n and $x \in$

$\bar{A}(\{i, j\}) = \bar{A}'(j)$. By the induction assumption (on $t = |T| - n$), $-x \in A_j' = A_j$.

The proof is now complete.

3. **N.D.S.s and faces.** The main object of this section is a proof of Theorem 5.

The proof of Theorem 5 relies on Lemmas 2, 3 and 4. Lemma 5 states that distinct 1-N.D.S.s are disjoint; it is presented on its own merit.

In the following, let A_T be a family of $|T| = n + t$ cones in R^n such that $A(T) = \{0\}$.

LEMMA 2. *If* $\dim \bar{A}(B) < |B| - k$ *and* $0 \leqslant k' \leqslant k$ *then* B *contains a* k'*-N.D.S. of* A_T *and* B *contains a* k'*-face of* A_T.

PROOF OF LEMMA 2. By the conditions of Lemma 2, $\dim \bar{A}(B) \leqslant |B| - k'$. Let B' be a minimal subset of B which satisfies $\dim \bar{A}(B) \leqslant |B'| - k'$. It is easily verified that B' is a k'-N.D.S. of A_T and, by Theorem 4, B' is a k'-face of A_T.

LEMMA 3. A. *If* B *is an N.D.S. of* A_T, $M \subset B$ *and* $\bar{A}(M)$ *contains an* $|M|$*-dimensional subspace, then* $M = \varnothing$.

B. *If* A_T *is an N.D.F.,* $B \subset M \subset T$, B *is a* k*-face of* A_T *and* $\bar{A}(M)$ *contains an* $(|M| - k)$*-dimensional subspace, then* $M = B$.

C. *If* A_T *is an N.D.F.,* $B_1 \subsetneq B_2 \subset T$ *and* B_i *is a* k_i*-face of* A_T *for* $i = 1, 2$, *then* $k_1 < k_2$.

PROOF OF LEMMA 3. Part A follows from Theorem 3 applied to the family $\{\bar{A}(B) \cap A_i : i \in B\}$ in span $\bar{A}(B)$. *Proof of* B: The case $B = T$ is trivial, assume therefore that B is a proper subset of T. Let Y be a subspace complementary to $\bar{A}(B)$ relative to R^n.

Define $C_{T\setminus B} = \{C_i : A_i \cap Y_i : i \in T \setminus B\}$, a family of cones in Y.

For each $S \subset T \setminus B$, $C(S) = A(S) \cap Y$, $\bar{C}(S) = \bar{A}(S \cup B) \cap Y$ and $\dim C(S) = \dim A(S) - \dim \bar{A}(B)$.

It follows that $C_{T\setminus B}$ is an N.D.F. in Y. Since $\bar{C}(M \setminus B) = \bar{A}(M) \cap Y$ and $\bar{A}(M)$ contains an $(|M| - k)$-dimensional subspace it follows that $\bar{C}(M \setminus B)$ contains an $[(|M| - k) - \dim \bar{A}(B) = |M| - k - (|B| - k) = |M - B|]$-dimensional subspace. By Theorem 3 applied to $C_{T\setminus B}$, $M \setminus B = \varnothing$, proving part B.

Part C is an immediate result of B.

LEMMA 4. *If* A_T *is an N.D.F. and* B *is a* k*-face of* A_T, *then* $0 \leqslant k \leqslant t$, $k = 0$ *iff* $B = \varnothing$, $k = t$ *iff* $B = T$.

Lemma 4 is easily derived from the definitions of a k-face, the definition of an N.D.F. and Theorem 3.

LEMMA 5. *If* B_1 *and* B_2 *are distinct* 1-*N.D.S.s of* A_T *then* B_1 *and* B_2 *are disjoint.*

PROOF. By Theorem 4, B_1 and B_2 are 1-faces of A_T. $B_1 \cap B_2$ is a face of A_T since $\bar{A}(B_1 \cap B_2) = \bar{A}(B_1) \cap \bar{A}(B_2)$ is a subspace.

Since B_1 and B_2 are distinct, $B_1 \cap B_2$ is either a proper subset of B_1 or a proper subset of B_2. Suppose that $B_1 \cap B_2$ is a proper subset of B_1. Since B_1 is a 1-N.D.S.,

$$|B_1 \cap B_2| - \dim \bar{A}(B_1 \cap B_2) < |B_1| - \dim \bar{A}(B_1) = 1.$$

Therefore $\dim \bar{A}(B_1 \cap B_2) \geq |B_1 \cap B_2|$ and, by Lemma 3.A, $B_1 \cap B_2 = \emptyset$, completing the proof.

PROOF OF THEOREM 5. Suppose that A_T is an N.D.F. in R^n. We have to prove that for any subset B of T, B is an N.D.S. of A_T iff B is a face of A_T.

If B is a k-N.D.S. then B is a k-face by Theorem 4.

We assume that B is a k-face and not a k-N.D.S. and derive a contradiction.

If B is not a k-N.D.S. of A_T then there exists a proper subset B' of B such that $|B'| - \dim \bar{A}(B') \geq B - \dim \bar{A}(B) = k$. Therefore $\dim \bar{A}(B') \leq |B'| - k$ and $k \geq 0$ by Lemma 4. By Lemma 2, B' contains a k-face C. Since C is a proper subset of B and A_T is an N.D.F., we have by Lemma 3.C that $k < k$, a contradiction. The proof of Theorem 5 is now complete.

4. **Remarks.** We will briefly discuss some results which may be obtained using properties of N.D.F.s:

1. Reconstructing dimensions of intersections of convex sets.

We can prove that if K_T and K'_T are two finite families of convex sets in R^n and $\dim \bigcap \{K_i: i \in S\} = \dim \bigcap \{K'_i: i \in S\}$ for each $S \subset T$ with $|S| \leq n + 1$, then $\dim \bigcap \{K_i: i \in T\} = \dim \bigcap \{K'_i: i \in T\}$ (see [3]).

2. In [4] we use properties of N.D.F.s and Gale diagrams (see [2, Chapter 5, §4], for Gale diagrams) to establish connections between N.D.F.s and convex polytopes:

For each N.D.F., $A_T = \{A_i: i \in T\}$ in R^n there is a $(|T| - n - 1)$-polytope P such that the lattice of faces of A_T ordered by the inclusion relation is isomorphic to the lattice of faces of P ordered by the inclusion relation.

This result enables one to obtain properties of families of cones by using well-known theorems on convex polytopes. An illustration is [5]:

3. Using N.D.F.s, [4], and properties of neighborly polytopes we can generalize a result of M. J. C. Baker [1] and prove a theorem which is equivalent to the following:

Let F be a finite family of at least $n + 1 + t$ 'convex' sets on S_n ($t \geq 0$). If every $n + 1$ members of F have nonempty intersection then there are $n + 1 + [t/2]$ members of F whose intersection is nonempty. (S_n is the n-dimensional unit sphere and a set is 'convex' if it is the intersection of a convex cone with apex 0 in R^{n+1} with S_n.)

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